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Note

Convergence Properties of the Finite-Element Method for Bénard Convection in an Infinite Layer

In any numerical approximation to a continuum problem, it is important to establish convergence to the exact solution as the number of discrete points used in the approximation is increased. In finite-element or finite-difference approximations, for example, this involves ensuring that grid-converged solutions are obtained as the mesh size h is decreased. For sufficiently small h, the error will vary as h to a certain power, which is the order of convergence. There is a considerable benefit in knowing the order of convergence of a particular method since it allows the extrapolation, to zero mesh size, of results obtained at small enough h. This approach has been used, for example, by Churchill *et al.* [1] and by Ozoe *et al.* [2].

The purpose of the present note is to investigate the convergence of the finite-element approximation, for the prediction of the critical Rayleigh number and cell size for the onset of Bénard convection in an infinite, horizontal layer.

In order to discuss convergence one must have a measure of the error in the approximate solution. For the finite-element method this is usually taken to be the mean square error in the first derivatives of the velocity and temperature, and the mean square error in the pressure. The rate of convergence achieved depends on the element used (the higher the degree of polynomial interpolation, the faster the rate of convergence). For example, the present study uses a nine-noded quadrilateral element, with biquadratic variation of the velocity and temperature, and linear discontinuous variation of the pressure; for this element the above errors converge as the square of the length of the longest element edge in the mesh.

If the solution to a problem has a simple bifurcation point then it can be shown [3] that the finite-element discretization also has one. Further, the rate of convergence of the critical parameter is the square of that for the fields: we shall say that the convergence of the critical parameter is superconvergent. For the element used here, this means that the error in the critical Rayleigh number converges as the fourth power of the mesh spacing h. On the other hand, for an element with bilinear interpolation for the velocity and temperature it converges only as h^2. It is clearly desirable to use higher-order interpolation in order to exploit this superconvergence at a simple bifurcation point. It should be noted that theoretical results on superconvergence for more complex singularities have not yet been proven.

To demonstrate the convergence of the finite-element method it is necessary to choose a problem with a known analytical solution, and the problem of Bénard

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convection in an infinite horizontal layer is an obvious choice. However, a precise computation of the critical Rayleigh number is not straightforward, and previous work has involved various assumptions. For example, Ozoe *et al.* [2] compute the Nusselt number at several values of the Rayleigh number just above critical, and extrapolate back to obtain the critical value for the onset of convection. However, with this technique they obtain a value of the critical Rayleigh number which agrees with the exact result to only two significant figures, even when extrapolated to zero mesh size.

For Bénard convection in a **finite** layer, Cliffe and Winters [4] computed the critical Rayleigh number using an algorithm which locates symmetry-breaking bifurcation points exactly (for a given grid). For an infinite layer, this approach can not be used since the cell width is not known a priori. We propose, therefore, a novel method which uses an algorithm for locating a higher-order singularity, namely the coalescence of two symmetry-breaking bifurcation points [5]. This allows us to predict both the critical Rayleigh number and the width of the cell. The convergence properties for the prediction of a coalescence point are not known, but might be expected to be superconvergent. This particular method has the added advantage of exploiting the symmetry of the problem so that the calculation is carried out over a quarter of the full domain. This makes it more certain that the asymptotic region of small mesh size, for which the fourth-power law is expected, will be achieved for a given cost.

To see why the critical Rayleigh number is a coalescence point we consider the problem shown in Fig. 1. For a given assumed value of the cell width, there will be a corresponding critical value of the Rayleigh number. As the cell width varies then the critical point will trace out a path, the neutral stability curve which is familiar

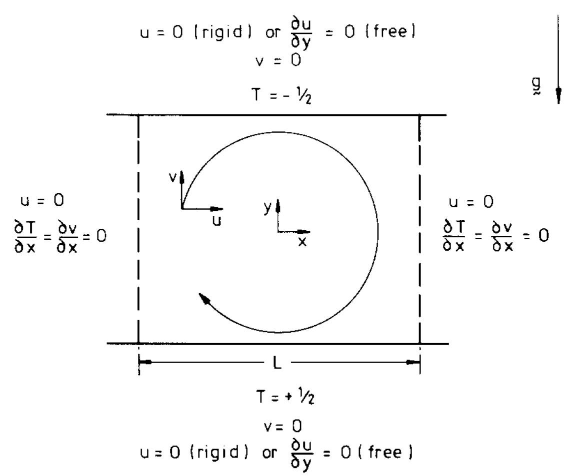

FIG. 1. The geometry and boundary conditions for one cell in the infinite Bénard problem.

from linear stability theory. This is shown in Fig. 2a. The minimum of this path gives the critical Rayleigh number and cell width for the onset of convection in an infinite layer. Now, if we consider the variation of the solution with cell width, for **fixed** Rayleigh number, then section A–B through Fig. 2a shows that this has the form of two pitchfork bifurcations, sketched in Fig. 2b. Clearly, as the Rayleigh number decreases these move closer together, and finally coalesce at the minimum of the neutral stability curve.

The algorithm for locating the coalescence point of two symmetry-breaking bifurcation points has been discussed in [5, 6]. The basic principle is to take the set of equations chosen to model the problem, and to add further equations which represent conditions satisfied at the point of coalescence. These additional equations typically involve first- and second-order derivatives of the original equations with respect to the parameters and the variables. This extended system of equations is solved by Newton's method, which converges to the point of coalescence for a sufficiently good initial guess.

We have applied the coalescence algorithm to the infinite Bénard problem with both free and rigid horizontal surfaces. In the case of free horizontal surfaces the analytical value of the critical Rayleigh number is $27\pi^4/4$ and the cell width is $\sqrt{2}$. The Navier–Stokes and energy equations were solved in the Boussinesq approximation [4], extended in the manner described in [5]. The equations are scaled so that the cell-width appears as an explicit parameter. The initial guess for the solution was taken to be the trivial, conducting solution at values of the Rayleigh number and cell-width close to the known critical point. The Newton iterations were found to converge rapidly, typically after three or four iterations. All calculations were carried out over a symmetric section of the cell, which is one quarter of the full domain. This section was discretized with a grid of N nine-noded quadrilateral elements in the horizontal and vertical directions, which we denote an $N \times N$ grid. The CPU times per iteration were 0.1, 0.4, 2.0, and 11.6 s for 2×2,

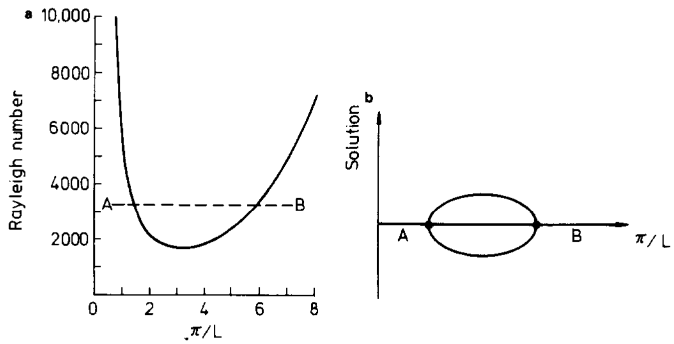

FIG. 2. (a) Neutral stability curve showing the variation of the critical Rayleigh number with wavenumber π/L, where L is the cell-width. (b) Section A–B through (a) showing the solution as a function of the wavenumber.

TABLE I

Critical Rayleigh Number for Free Horizontal Surfaces

Grid	Rayleigh number	Cell-width
2×2	657.41318836	1.4154766916
4×4	657.54081120	1.4142358838
8×8	657.51385394	1.4142139219
16×16	657.51153060	1.4142135680
Extrap	657.51137571	
Exact	657.51136446	1.4142135624

4×4, 8×8, and 16×16 grids respectively, on a Cray–1S. Note that the finest 16×16 grid is equivalent to 32×32 elements (or 65×65 nodes) over the full domain of the cell.

Table I shows the predicted values of our finite-element calculation for successively finer grids, in the case of free horizontal surfaces. The result of extrapolation of the values for the two finest grids, according to an h^4 law, is also given. The extrapolated value of the critical Rayleigh number is seen to agree with

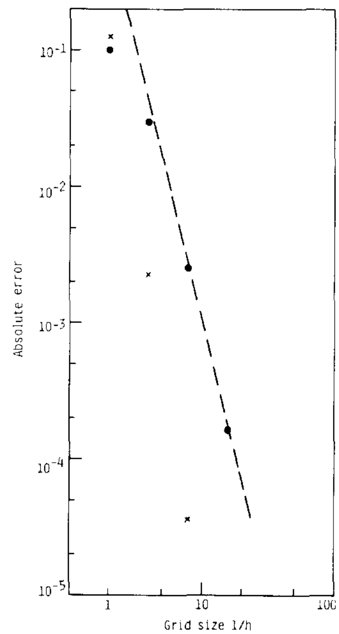

FIG. 3. The absolute error in (●) the predicted critical Rayleigh number and (×) the predicted cell-width $L(\times 100)$ as a function of N, the number of elements in the vertical or horizontal directions. The dashed line shows an N^{-4} variation.

the exact, analytical result to seven significant figures. It is interesting that the predicted values of the cell-width convergence even more rapidly, and the result for the finest grid is correct to eight significant figures, without extrapolation. Fig. 3 shows the log of the error plotted against the log of N, and this confirms the h^4 dependence. The error in the predicted cell width is also plotted and this is found to converge at a rate close to h^6. Further theoretical work is needed to clarify this enhanced superconvergence of the critical cell width.

Table II shows the predicted values for rigid surfaces. As before, the extrapolated value of the critical Rayleigh number agrees with the exact result to seven significant figures.

The superconvergence of the critical Rayleigh number for the computation of the coalescence point is consistent with the theoretical work on simple bifurcation points [3]. One way of interpreting this is to compare it to the Rayleigh–Ritz method of estimating eigenvalues, in which the error in the eigenvalues is the square of the error in the eigenvector.

The convergence properties of the finite-element method depend on the element used and also on the smoothness of the solution to the problem. For example, sharp corners in the solution domain can reduce the rate of convergence because the solution may not be sufficiently smooth for the optimal rate to apply. However, the effect of sharp corners is localized and it is possible that some form of mesh refinement, together with the use of a measure of the error which is weighted so as to reduce the effect of the corner, may result in the optimal convergence rate being recovered. This is an important area for further theoretical research.

In summary, we have proposed a new method for predicting the critical Rayleigh number and cell-width for Bénard convection in an infinite horizontal layer. We have used this to demonstrate the superconvergence of the finite-element method for the prediction of a point of coalescence of two symmetry-breaking bifurcations. Extrapolation of our predictions gives a value of the critical Rayleigh number which is exact to seven significant figures.

TABLE II

Critical Rayleigh Number for Rigid Horizontal Surfaces

Grid	Rayleigh number	Cell-width
2×2	1715.3843320	1.0094390241
4×4	1708.3822333	1.0081118707
8×8	1707.8034196	1.0081072657
16×16	1707.7644279	1.0081085142
Extrap	1707.7618285	
Exact	1707.762	1.008

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